The Laplace Transform

* **Preface:-** The theory of laplace transforms or Laplace transformation also referred to as operational Calculus, has in recent years become an essential part of the mathematical background required of engineers, physicists, mathematicians, and other scientists. This is because, in addition to being of great theoritical interest in itself, laplace transform methods provide easy and effective means for the solution of many problems arising in various fields of science and engineering.

* **Definition of the laplace Transform:-**

Let $f(t)$ be a function of t specified for $t>0$. Then the laplace transform of $f(t)$, denoted by $\mathcal{L}f(t)$, is defined by:

$$\mathcal{L}f(t) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

where s is a complex number $s = \sigma \pm j\omega$

* **Laplace Transforms of some elementary functions:-**

	$f(t)$	$F(s)$	
1.	$\delta(t)$	1, $s>0$	Unit impulse function
2.	1	$\frac{1}{s}$, $s>0$	Unit-step function.
3.	t	$\frac{1}{s^2}$, $s>0$	Unit Ramp function
4.	e^{at}	$\frac{1}{s-a}$, $s>a$	exponential function
5.	t^n	$\frac{n!}{s^{n+1}}$, $s>0$	
6.	$\sin(at)$	$\frac{a}{s^2+a^2}$, $s>0$	
7.	$\cos(at)$	$\frac{s}{s^2+a^2}$, $s>0$	

Therom 1: if $f(t)$ is sectionally continuous in every finite interval $0\le t\le N$ and of exponential order, then its laplace transforms $F(s)$ exists.

* **Some Important properties of laplace Transform:-**

1. Linearity property: $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} = c_1F_1(s) + c_2F_2(s)$

Ex: $\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} = 4\mathcal{L}(t^2) - 3\mathcal{L}(\cos 2t) + 5\mathcal{L}(e^{-t})$

$$= 4\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right) + 5\left(\frac{1}{s+1}\right)$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

2. First shifting property: if $\mathcal{L}\, f(t) = F(s)$ then $\mathcal{L}\, e^{at} f(t) = F(s-a)$

Ex:- Since $\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}$ then $\mathcal{L}\{e^{-t}\cos(2t)\} = \frac{(s+1)}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$

3. change of scale property: if $\mathcal{L}\, f(t) = F(s)$ then $\mathcal{L}\, f(at) = \frac{1}{a} F(\frac{s}{a})$

Ex:- $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, then $\mathcal{L}\, \sin(3t) = \frac{1}{3} \cdot \frac{1}{(\frac{s}{3})^2+1} = \frac{3}{s^2+9}$

4. Laplace transform of derivatives:

if $\mathcal{L}\, f(t) = F(s)$ then, $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

Ex: if $f(t) = \cos 3t$ then $F(s) = \frac{s}{s^2+9}$ and we have

$\mathcal{L}\{f'(t)\} = \mathcal{L}\{-3\sin 3t\} = s\left(\frac{s}{s^2+9}\right) - 1 = \frac{-9}{s^2+9}$

5. laplace transform of integrals:-

if $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$

Ex: Since $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$, then we have $\mathcal{L}\left\{\int_0^t \sin 2u\,du\right\} = \frac{2}{s(s^2+4)}$

6. Multiplication by t^n:- if $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) = (-1)^n \frac{d^n}{ds^n} F(s)$$

Ex:- Since $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$, we have

$$\mathcal{L}\{t e^{2t}\} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$

$$\mathcal{L}\{t^2 e^{2t}\} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$$

7. Division by t:- if $\mathcal{L}\{f(t)\} = F(s)$, then

$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(u)\,du$, provided that $\lim_{t\to 0} f(t)/t$ exists

Ex: Since $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ and $\lim_{t\to 0} \frac{\sin t}{t} = 1$, we've

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{du}{u^2+1} = \tan^{-1}\left(\frac{1}{s}\right)$$

8. Initial-value theorem:- $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Ex:- $\lim_{t\to 0} \sin(at) = 0 = \lim_{s\to\infty} s.\frac{a}{s^2+a^2} = \lim_{s\to\infty} \frac{a/s}{1+\frac{a^2}{s^2}} = \frac{0}{1} = 0$

9. Final-value theorem:- $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Ex:- $\lim_{t\to\infty} t = \infty = \lim_{s\to 0} s.F(s) = \lim_{s\to 0} s.\frac{1}{s^2} = \lim_{s\to 0}\frac{1}{s} = \infty$

* Methods of finding laplace Transforms:-

1. Direct method: This involves direct use of definition $F(s) = \int_0^\infty e^{-st} f(t)\,dt$.
2. Series method: if $f(t)$ has a power series expansion given by

$$f(t) = a_0 + a_1 t + a_2 t^2 + \dots = \sum_{n=0}^{\infty} a_n t^n$$

then $\mathcal{L}\{f(t)\} = F(s) = \sum_{n=0}^{\infty} \frac{n!\,a_n}{s^{n+1}}$

3. Method of differential Equations: This involves finding a differential Equation satisfied by $f(t)$ and the using the given theorems to solve.
4. Use Laplace Tables.

(Proplems)

1. prove that (a) $\mathcal{L}(1) = \frac{1}{s}, s>0$; (b) $\mathcal{L}(t) = \frac{1}{s^2}, s>0$, (c) $\mathcal{L}e^{at} = \frac{1}{s-a}, s>a$.
2. Find the laplace transforms of each of the following functions:
   (a) $2e^{4t}$ Ans. $\frac{2}{(s-4)}, s>4$.
   (b) $5t-3$ Ans. $\frac{(5-3s)}{s^2}, s>0$.
   (c) $2t^2 - e^{-t}$ Ans. $(4+4s-s^3)/s^3(s+1), s>0$.
   (d) $3\cos 5t$ Ans. $3s/(s^2+25), s>0$.
   (e) $6\sin 2t - 5\cos 2t$ Ans. $(12-5s)/(s^2+4), s>0$.
3. Find $\mathcal{L}\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$ Ans. $\frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}$
4. Evalute each of the following:
   (a) $\mathcal{L}\,t^3 e^{-3t}$ Ans. $6/(s+3)^4$, (b) $\mathcal{L}\{2e^{3t}\sin 4t\}$ Ans. $8/(s^2-6s+25)$

5. Given $F(t) = \begin{cases} 2t & 0 \le t \le 1 \\ t & t > 1 \end{cases}$ find (a) $\mathcal{L}\{F(t)\}$, b. find $\mathcal{L}\{F'(t)\}$.

Ans. (a) $\frac{2}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}$, (b) $\frac{2}{s} - \frac{e^{-s}}{s}$.

6. verify directly that $\mathcal{L}\left\{\int_0^t (u^2 - u + e^{-u})\,du\right\} = \frac{1}{s}\mathcal{L}\{t^2 - t + e^{-t}\}$.

7. show that $\mathcal{L}\{t^2 \sin t\} = \frac{6s^2 - 2}{(s^2+1)^3}$.

8. show that $\mathcal{L}\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \ln\left(\frac{s+b}{s+a}\right)$

9. verify the initial-value theorem for the functions (a) $3 - 2\cos t$, (b) $(2t+3)^2$

10. verify the final-value theorem for the functions (a) $t^3 e^{-2t}$, (b) $1 + e^{-t}(\sin t + \cos t)$.

The Inverse Laplace Transform

*** Definition of Inverse Laplace Transform:-**

If the Laplace transform of a function $f(t)$ is $F(s)$ then $f(t) = \mathcal{L}^{-1} F(s)$.

ex: Since $\mathcal{L}\, e^{-3t} = \frac{1}{s+3}$ then $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$.

*** Some Inverse laplace Transforms:**

	$F(s)$	$\mathcal{L}^{-1}F(s) = f(t)$
1.	1	$\delta(t)$ Unit Impulse function
2.	$\frac{1}{s}$	1 Unit step function.
3.	$\frac{1}{s^2}$	t Ramp function.
4.	$\frac{1}{s-a}$	e^{at} exponential function.
5.	$\frac{1}{s^{n+1}}$ $n=0,1,2,\dots$	$\frac{t^n}{n!}$
6.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
7.	$\frac{s}{(s^2+a^2)}$	$\cos at$

*** Some Important Properties of Inverse Laplace Transforms:-**

1. Linearity property: $\mathcal{L}^{-1}\{c_1 F_1(s) + c_2 F_2(s)\} = c_1 \mathcal{L}^{-1}\{F_1(s)\} + c_2 \mathcal{L}^{-1}\{F_2(s)\}$

$$= c_1 f_1(t) + c_2 f_2(t)$$

ex: $\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

2. First shifting property: If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

ex: Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, we have $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = \frac{1}{2}e^{t}\sin 2t$

3. Change of scale property: If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)$

ex: Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t$, we have:

$$\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \frac{1}{2}\cos\frac{4t}{2} = \frac{1}{2}\cos 2t.$$

4. Inverse laplace transform of derivatives: if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then;

$$\mathcal{L}^{-1}\{F^{(n)}(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}F(s)\right\} = (-1)^n t^n f(t)$$

ex: since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{-2s}{(s^2+1)^2}$, we have

$$\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} = -t\sin t$$

5. Inverse laplace transform of Integrals: if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then

$$\mathcal{L}^{-1}\left\{\int_s^\infty F(u)\,du\right\} = \frac{f(t)}{t}$$

ex: Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$, we have

$$\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)du\right\} = \mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\} = \frac{1-e^{-t}}{t}$$

6. Multiplication by s^n if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $f(0) = 0$ then

$$\mathcal{L}^{-1}\{s\,F(s)\} = f'(t)$$

ex: since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\sin 0 = 0$ then

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)}\right\} = \frac{d}{dt}(\sin t) = \cos t$$

7. Division by s ; if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$$

ex: Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\,du = \frac{1}{4}(1-\cos^2 t)$$

* Methods of finding Inverse laplace transforms:

1. Partial fractions method: Any rational function $P(s)/Q(s)$ where $P(s)$ and $Q(s)$ are polynomials, with the degree of $P(s)$ less than of $Q(s)$, can be written as the sum of rational functions called partial fraction having the form $\frac{A}{(as+b)^r}$, $\frac{As+B}{(as^2+bs+c)^r}$, ... where $r = 1, 2, 3, \ldots$ By finding the inverse Laplace transform of each of the partial fraction, we can find $\mathcal{L}^{-1}\{P(s)/Q(s)\}$.

Ex: $\frac{2s-5}{(3s-4)(2s+1)^3} = \frac{A}{(3s-4)} + \frac{B}{(2s+1)^3} + \frac{C}{(2s+1)^2} + \frac{D}{(2s+1)}$

Ex: $\frac{3s^2-4s+2}{(s^2+2s+4)^2(s-5)} = \frac{As+B}{(s^2+2s+4)^2} + \frac{Cs+D}{(s^2+2s+4)} + \frac{E}{(s-5)}$

The constants A, B, C ---etc, can be found by clearing of fractions and equating of like powers of s on both sides of the resulting equation or by using special methods.

2. Series methods. if $F(s)$ has a series expansion in inverse powers of s given by $F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \frac{a_3}{s^4} + \cdots$ then under suitable conditions we can invert term by term to obtain

$$f(t) = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \cdots$$

3- Method of differential equations.

4- Use of Tables.

(Problems & some Solved Examples)

1. Find each of the following inverse laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{4}{s-2}\right\}$, (c) $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$, (d) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}$

Ans. (a) $\frac{\sin 3t}{3}$, (b) $4e^{2t}$, (c) $t^3/3! = t^3/6$, (d) $\cos\sqrt{2}\,t$

2. Find $\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{s^2} + \frac{4}{s^3} - \frac{2s}{s^2+9} + \frac{18}{s^2+9}\right\}$

$$= 5t + 4\left(t^2/2!\right) - 2\cos 3t + 18\left(\tfrac{1}{3}\sin 3t\right) = 5t + 2t^2 - 2\cos 3t + 6\sin 3t.$$

3. prove that $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$

Ans. Since $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ then $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$F(s-a) = \int_0^\infty e^{-st}\left\{e^{at} f(t)\right\} dt = \mathcal{L}\left\{e^{at} f(t)\right\}$

Hence $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$.

4. Find each of the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$ Ans. $2e^{2t}(3\cos 4t+\sin 4t)$.

Hint: $\mathcal{L}^{-1}\left\{\dfrac{6(s-2)+8}{(s-2)^2+16}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s+12}{s^2+8s+16}\right\}$ Ans. $4e^{-4t}(1-t)$.

Hint: $\mathcal{L}^{-1}\left\{\dfrac{4(s+4)-4}{(s+4)^2}\right\}$

5. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$ Hint: $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}=\sin t$, $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\}=\int_0^t \sin u\,du = 1-\cos t$

repeat for $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$ to obtain Ans: $\dfrac{t^2}{2}+\cos t-1$

6. Apply Partial Fractions Method to find $\mathcal{L}^{-1}$ of the followings:-

(a) $\mathcal{L}^{-1}\left\{\dfrac{3s+16}{s^2-s-6}\right\}$, Ans. $5e^{3t}-2e^{-2t}$.

(b) $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{s^3-s}\right\}$, Ans. $1-\frac{3}{2}e^{-t}+\frac{1}{2}e^{t}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}\right\}$, Ans. $5e^{2t}-1.5e^{t/2}+2e^{-t}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{s^3+16s-24}{s^4+20s^2+64}\right\}$, Ans. $\frac{1}{2}\sin 4t+\cos 2t-\sin 2t$

(e) $\mathcal{L}^{-1}\left\{\dfrac{s^2-2s+3}{(s-1)^2(s+1)}\right\}$, As. $0.5(2t-1)e^{t}+1.5e^{-t}$